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Stationary RG flow and thermodynamics of conformal field theories*

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Abstract: We argue that one can understand the relationship between critical free-energy densities of theories connected via non-perturbative renormalization group flows, in terms of the the number of fields coupled to the class of different theories which flow into the same stationary renormalization group trajectory. As an explicit example, we study the free-energy density of bosonic and fermionic theories possessing strongly coupled critical points in $D = 3$. We construct a stationary trajectory which interpolates between the free massless theory of N scalars and a class of interacting theories including both bosons and fermions. The free-energy along this trajectory remains constant, but the degrees of freedom coupled to the underlying theories which flow into it are different. The difference in the degrees of freedom underlying two distinct points of the stationary trajectory, provides a natural explanation for the rational relationship between the critical free-energy densities of the free massless theory of N scalars and the $\mathrm{O}(N)$ vector model.

Keywords: Field Theories in Lower Dimensions, Nonperturbative Effects, Conformal and W Symmetry.

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Contents

1. Introduction

The recent progress in our understanding of the microscopic origin of the thermodynamic properties of black holes has sparked a renewed interest in the investigation of conformal field theories (CFTs) at finite temperature. There have been a number of calculations [1]–[5] regarding the free-energy density of CFTs emerging from AdS/CFT correspondence [6, 7, 8]. These are particularly interesting, as the AdS/CFT correspondence probes the strong-coupling regime of $\mathcal{N} = 4$ SYM$_4$ for large-N. The free-energy obtained is found to be 3/4 times the free-energy density of the same theory at its free-field limit. Such a result still lacks a proper understanding since it is related to non-trivial D-dimensional conformal field theories, where $D > 2$, which are not well understood.

It has also been suggested in [9], following earlier work of [10]–[14], that the free-energy density at criticality encodes important information for the RG-flow between different fixed points.[1] In particular, it has been claimed in [9] that studies of the free-energy density yield constraints regarding the low-energy spectrum of theories undergoing symmetry-breaking transitions.

To get more insight into the significance of the free-energy density at criticality, we elaborate here on some remarkable results regarding the three-dimensional O(N)

[1]For a different approach see [15].

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vector model and its fermionic counterpart, the Gross-Neveu model. These models have non-trivial critical points at large-N which are strongly coupled and represent a symmetry-breaking transition, making them important testing ground. A remarkable result is then that the leading-N free-energy density of the bosonic theory at its non-trivial critical point is $4/5$ times the free-energy density of the theory of N massless free scalars [12].

Our purpose here is to provide some physical insight into the origin of this simple relationship between the two critical points. To this end, we construct a theory of coupled bosonic and fermionic fields. Then, on the space of couplings we identify a stationary trajectory which interpolates between the free massless theory of N scalars and a class of interactive critical theories with an $\mathrm{O}(N/4)$ symmetry. The free-energy along this trajectory does not change. Its value remains equal to the free-energy of the theory of N massless free scalars. At one end of the trajectory only four $\mathrm{O}(N/4)$ massless fields contribute and therefore the underlying theory is that of N massless free scalars. We then find a point on the trajectory at which the underlying theory is that of five copies of the $\mathrm{O}(N/4)$ vector model. This result explains why the free-energy density at the non-trivial critical point of the $\mathrm{O}(N)$ vector model is $4/5$ times the free-energy density of the free theory of N massless scalars. Our method is quite general and could in principle be used to studying other theories which have strongly coupled critical points, such as $\mathcal{N} = 4$ SYM_4 in the large-N limit.

We organize our discussion as follows. In section 2, we introduce the free-energy density and briefly discuss its properties. In section 3, we review the salient features of the $\mathrm{O}(N)$ vector model as well as the fermionic Gross-Neveu model. Section 4 contains our main calculation. We construct an interactive field theory in three dimensions, making use of the models in section 3. We show that this theory possesses stationary trajectories along which the free-energy does not change. We identify a point on one of the trajectories whose free-energy is manifestly $5/4$ times the free-energy of the non-trivial critical point of the $\mathrm{O}(N)$ vector model. Finally, in section 5, we present our conclusions.

2. General remarks on the free-energy density

Consider a euclidean field theory in $S^1 \times \mathbb{R}^{d-1}$ geometry with periodic boundary conditions. Such a theory is related to a d-dimensional statistical mechanical system in thermal equilibrium, as the lattice spacing $a \to 0$. The free-energy density (free-energy per unit volume), for the latter system can be written as [11]

$$f(L, \tilde{g}, a) = f_0(\tilde{g}, a) - \frac{1}{L^d} C(L, \tilde{g}, a)\,, \tag{2.1}$$

where the temperature is $T = 1/L$ and $\tilde{g}$ denotes collectively the bare coupling constants. The term f_0 is the free-energy density at zero temperature, or equivalently the "bulk" free-energy density.

Under the basic assumption that in the limit $a \to 0$ the statistical mechanical system is described by a renormalizable QFT, UV renormalization of f_0 suffices to render eq. (2.1) finite. Furthermore, one can write

$$C(L, \tilde{g}, a) \equiv C_R(g_R, \mu L)\,, \tag{2.2}$$

where g_R are dimensionless coupling constants and μ is a mass scale. The subscript R denotes the renormalized quantities. It is important to stress that the dimensionless quantity $l = \mu L$ is *not* the standard RG scale since L^{-1} is a physical mass scale not related to the UV cut-off. Nevertheless, as C_R is an observable it should not depend on μ and should therefore satisfy the following RG equation (we drop the R subscript since we shall henceforth deal only with renormalized physical quantities)

$$\left(l\frac{\partial}{\partial l} + \beta(g)\frac{\partial}{\partial g}\right) C(g, l) = 0\,, \tag{2.3}$$

$$\beta(g) = \mu\frac{\partial g(l)}{\partial \mu}\,. \tag{2.4}$$

Eq. (2.3) is solved [11] by the introduction of the inverse correlation length $\xi^{-1}(g, l) = M(g, l)$ which has dimensions of mass and obeys the following RG equation

$$\left(l\frac{\partial}{\partial l} + \beta(g)\frac{\partial}{\partial g}\right) M(g, l) = 0\,. \tag{2.5}$$

Then, from eq. (2.3) and eq.(2.5) we may write

$$C(g, l) \equiv C(ML)\,. \tag{2.6}$$

Denoting $t = ML$, then $C(t)$ is a RG-invariant quantity.

The critical points of the theory correspond to certain values g_* of the coupling(s) such that

$$\beta(g_*) = 0\,. \tag{2.7}$$

From eq. (2.5), this implies the existence of a critical inverse correlation length $M_* = M(g_*)$ and therefore of a critical value for $C(t_*)$, $t_* = M_* L$. It can then be seen from eq. (2.3) and eq. (2.5) that the possible values of the critical correlation length - which would correspond to possible critical points of the theory - are given by the stationary points of $C(t)$.

If eq. (2.7) has more than one solutions, they must correspond to different critical points. The introduction of the length scale L, being essential for the existence of more that one stationary points of $C(t)$, provides the means of connecting the two theories flowing to the different critical points. These two critical points may in general have different massless degrees of freedom coupled into them and one might be tempted to view them as UV and IR critical points of the *same* theory. However, it is rather difficult to demonstrate the existence of such a RG flow as it is in principle non-perturbative. An explicit example of such a case is provided by the O(N) ϕ^4 theory and the O(N) Vector Model in $d = 3$. The latter theory has a non-trivial UV fixed point which has been argued to be the same as the IR fixed point of the former theory. This will be discussed in section 3.

Nevertheless, eq. (2.3) can still provide important insight towards the understanding of the RG flow between critical points having different massless degrees of freedom coupled into them. The essential observation is that eq. (2.3) allows the construction of stationary flows connecting theories with different field content. Indeed, had we been able to construct a function $C[t(t')]$ which is stationary for all values of t' in a certain interval, then, from eq. (2.7), this would correspond to a line of critical points parametrized by the new parameter t'. To all these critical points corresponds the same value of C, which is essentially a constant, therefore they have the same massless degrees of freedom coupled into them. In other words, t' now parametrizes different underlying theories flowing to the same critical point. This way we are able to deduce simple relations between the degrees of freedom coupled at the critical points of different theories.

We mentioned above that the quantity $C(g,l)$ carries information for all the degrees of freedom coupled to the theory at a certain RG-scale. The reason is that $C(g,l)$ is related to the energy momentum tensor. Consider a system in a "slab" of length L with partition function

$$Z_L = \int \mathcal{D}\phi \; e^{-S_L(\phi)} = e^{-F(L)} \,, \tag{2.8}$$

where $F(L)$ is the free-energy normalized so that

$$F(\infty) = 0 \,, \tag{2.9}$$

Under the coordinate change,

$$x_\mu \to x_\mu + \alpha_\mu(x) \,, \qquad \alpha_\mu(x) = \left[\left(1 + \frac{\Delta L}{L}\right) x_0 \,, \bar{x} \right] , \qquad \bar{x} = (x_1, \ldots, x_{d-1}) \tag{2.10}$$

we have

$$S_L \to S_L + \frac{\Delta L}{L} \int_{\text{slab}} \mathrm{d}^d x \, T_{00}(x) \,. \tag{2.11}$$

If the partition function is at a fixed point of the RG, it remains invariant under such a change. This gives to leading order in ΔL

$$Z_L = e^{-F(L)} = e^{-F(L) - \Delta L \frac{\partial F}{\partial L} - \frac{\Delta L}{L} \int_{\text{slab}} \mathrm{d}^d x \langle T_{00}(x) \rangle} \,, \tag{2.12}$$

which gives the change of the free-energy in terms of the T_{00}-component of the energy momentum tensor as [10]

$$L \frac{\partial F}{\partial L} = - \int_{\text{slab}} \mathrm{d}^d x \langle T_{00}(x) \rangle \,. \tag{2.13}$$

Using eq. (2.1), we then obtain

$$\langle T_{00}\rangle = (1-d)\frac{C(g,l)}{L^d} + \frac{1}{L^d}\beta(g)\frac{\partial}{\partial g}C(g,l)\,. \tag{2.14}$$

Relations such as eq. (2.14), acquire their full power in two dimensions where they lead to the important result that $C(g,l)$ at a critical point is proportional to the central charge of the corresponding CFT.

3. Critical bosonic and fermionic theories in $D = 3$

In this section, we review $\mathrm{O}(N)$ bosonic and $\mathrm{U}(N)$ fermionic models in the large-N limit.

3.1 The $\mathrm{O}(N)$ vector model

The lagrangean density for the $\mathrm{O}(N)$ bosonic vector model is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} + V(|\vec{\phi}|)\,, \tag{3.1}$$

where $\vec{\phi}(x) = \phi^a(x)$, $a = 1, 2, \ldots, N$ is the $\mathrm{O}(N)$-vector scalar field, and $V(|\vec{\phi}|)$ is an interaction potential whose explicit form is not important for our purposes. The partition function in the large-N limit exhibits a universal behavior. This has been studied in the case when

$$V(|\vec{\phi}|) = \lambda(\vec{\phi}\cdot\vec{\phi})^2 \tag{3.2}$$

and also when the vector field is constrained by

$$\vec{\phi}\cdot\vec{\phi} = 1\,. \tag{3.3}$$

In the latter case, the lagrangean density may be written in terms of an auxiliary scalar field σ as

$$\mathcal{L} = \frac{1}{2}\,\vec{\phi}\cdot(-\partial^2+\sigma)\vec{\phi} + \frac{N}{g_0}\sigma\,. \tag{3.4}$$

This theory possesses a non-trivial critical point at each order in a $1/N$ expansion, as the coupling constant is kept fixed [16, 17]. This critical point is UV-stable, corresponds to a non-trivial three-dimensional CFT and represents the $\mathrm{O}(N) \to \mathrm{O}(N-1)$ symmetry breaking transition. The critical theory is strongly coupled, since the $1/N$ expansion corresponds to a resummation of an infinite number of diagrams.

In the spirit of the suggestion in [9], we can measure the massless degrees of freedom coupled to a critical theory if we heat it up to a temperature $T = 1/L$ where we impose periodic boundary conditions along euclidean time. This is equivalent to placing the theory in a slab with one finite dimension of length L playing the role of inverse temperature. Starting from the lagrangean (3.4) and using a saddle point approximation, we obtain the free energy density

$$f(L, g_0) = \frac{N}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^2\bar{p}}{(2\pi)^2} \ln(\bar{p}^2 + \omega_n^2 + \sigma_s) - \frac{N}{g_0}\,\sigma_s\,, \tag{3.5}$$

where $\bar{p} = (p_1, p_2)$ and σ_s is the saddle-point value of the auxiliary field σ constrained by the gap equation

$$\frac{\partial f}{\partial \sigma_s} = \frac{N}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^2\bar{p}}{(2\pi)^2} \frac{1}{\bar{p}^2 + \omega_n^2 + \sigma_s} - \frac{N}{g_0} = 0\,. \tag{3.6}$$

The bare coupling constant g_0 is renormalized by the RGE

$$\frac{1}{g_R} = \frac{1}{g_0} - \frac{1}{2} \int_\Lambda \frac{d^3p}{(2\pi)^3} \frac{1}{p^2}\,, \tag{3.7}$$

where we introduced the cutoff Λ to regulate the divergent integral. Then the quantity $f(L) - f(0)$ is finite, and can be written in terms of the renormalized coupling constant g_R via eqs. (3.6) and (3.7).

At a critical point, we have $1/g_R \to 0$. In that case, the gap equation (3.6) becomes

$$\frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^2\bar{p}}{(2\pi)^2} \frac{1}{\bar{p}^2 + \omega_n^2 + \sigma_s} - \frac{1}{2} \int_\Lambda \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} = 0\,, \tag{3.8}$$

which has a non-trivial solution, $\sigma_s = m_*^2$, where

$$m_* L = 2\ln\tau\,, \qquad \tau = \frac{1+\sqrt{5}}{2}\,. \tag{3.9}$$

At this critical point, the O(N) vector model coupling stays at its zero temperature critical value g_*, (this is sometimes referred to as the "finite-size scaling regime" [18]). One then obtains the remarkable result [12]

$$\begin{aligned}
\frac{C(g_*)}{N} &= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \ln p^2 - \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^2\bar{p}}{(2\pi)^2} \ln(\bar{p}^2 + \omega_n^2 + m_*^2) + \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{m_*^3}{p^2} \\
&= \frac{1}{2\pi L^3} \left[\mathrm{Li}_3(e^{-m_*L}) - \ln(e^{-m_*L}) \mathrm{Li}_2(e^{-m_*L}) + \frac{1}{6}\,(m_*L)^3 \right] \\
&= \frac{4}{5} \frac{\zeta(3)}{2\pi L^3}\,.
\end{aligned} \tag{3.10}$$

The simplicity of this result is due to some remarkable identities for the dilogarithm and the trilogarithm at the special point eq. (3.9) [19] (See appendix).

As the free-energy density of a theory of N massless free scalars is

$$\frac{C_0(g_0)}{N} = \frac{\zeta(3)}{2\pi}\,, \tag{3.11}$$

one is tempted to conclude from eqs. (3.10) and (3.11) that, for large-N, the number of massless degrees of freedom coupled to the non-trivial fixed-point of the O(N) vector models, is 4/5 times the massless degrees of freedom coupled to the free-field theory fixed-point. However, the strongly coupled critical theory is not related in an entirely obvious way to the free-field theory of N massless scalars. It has been argued [16] that the critical UV-stable fixed point of the O(N) vector model is in the same universality class with the IR-stable fixed point of the O(N) invariant ϕ^4 theory in $2 < d < 4$. Such a correspondence is only valid at the critical regime when one can disregard an infinite number of irrelevant operators. Therefore, the RG-flow from the free-field theory of N massless bosons, (corresponding to the UV-stable fixed point of the O(N) invariant ϕ^4 theory), to the IR-stable fixed point is non-perturbative. This means essentially that had we been able to obtain the full solution of the RG-equations, the massless free theory would correspond to the weak-coupling limit and the non-trivial critical theory eq. (3.10) to the strong coupling limit.

3.2 The U(N) Gross-Neveu model

Another three-dimensional theory with a strongly coupled fixed point is the U(N) Gross-Neveu model [16, 17]. Its lagrangean is written is terms of an auxiliary scalar field λ as

$$\mathcal{L} = -\bar{\psi}^\alpha(\gamma_\mu\partial^\mu + \lambda)\psi^\alpha + \frac{N}{2G}\lambda^2\,, \tag{3.12}$$

where $\alpha = 1, 2, \ldots, N$.[2] This theory has a UV-stable critical point at each order in the $1/N$ expansion as the coupling G is kept fixed. This critical point represents a space-parity breaking transition.

Placing the theory in a slab with one finite dimension of length L amounts to imposing antiperiodic boundary conditions for the fermions along euclidean time. Then, when the coupling stays at its bulk critical value G_* the gap equation gives zero expectation value for λ. This is a consequence of the absence of zero modes for fermions and antiperiodic boundary conditions. It essentially means that $C(G_*)$ for this model is given by the free-field theory result [20] and not by a complicated relation involving polylogarithms such as eq. (3.10). Therefore, for some time its was thought that nothing interesting happens in three-dimensional fermionic models at the finite-size scaling regime.

However, it was recently discovered [21, 14] that certain three-dimensional fermionic models do exhibit non-trivial behavior in the finite-size scaling regime. The theory discussed in [14] is the U(N) Gross-Neveu model for fixed total fermion number B. The fixed fermion number constraint is introduced into the theory via a delta

[2] For the euclidean three-dimensional gamma matrices we use the hermitean two-dimensional representation $\gamma_1 = \sigma^1$, $\gamma_2 = \sigma^2$, $\gamma_3 \equiv \gamma_0 = \sigma^3$, with σ^i, $i = 1, 2, 3$ the usual Pauli matrices.

function $\delta(\hat{N} - B)$, where

$$\hat{N} = \int \mathrm{d}^2\bar{x}\psi^\dagger(\bar{x})\psi(\bar{x}) \,, \qquad \bar{x} = (x_1, x_2) \tag{3.13}$$

is the fermion number operator. Using an auxiliary scalar field $\theta(x_3)$ to impose the delta function constraint, the action for this model is written in the slab geometry

$$\mathcal{S} = -\int_{\text{slab}} \bar{\psi}^\alpha(\gamma_\mu\partial^\mu + \mathrm{i}\gamma_3\theta + \lambda)\psi^\alpha + \frac{N}{2G}\int_{\text{slab}} \lambda^2 + \mathrm{i}B\int_{\text{slab}} \theta \,. \tag{3.14}$$

It was shown in [14] that for certain values of θ the gap equation of the model yields a non-zero expectation value for λ when the coupling stays at the bulk critical value G_*. Consequently, $C(G_*)$ is no longer given by the free field theory result but by a non-trivial expression involving polylogarithms as

$$\begin{aligned} \frac{C(G_*)}{NL^3} &= -\frac{\langle\lambda\rangle^3}{6\pi} + \frac{1}{\pi L^3}\left[\ln\left(e^{-L\langle\lambda\rangle}\right)\mathrm{Li}_2\left(-e^{-L\langle\lambda\rangle}, L\langle\theta\rangle\right) - \mathrm{Li}_3\left(-e^{-L\langle\lambda\rangle}, L\langle\theta\rangle\right)\right] + \\ &\quad + \frac{\langle\theta\rangle}{2\pi L^3}\left[Cl_2(2\phi) - Cl_2(2\phi - 2L\langle\theta\rangle) - Cl_2(2L\langle\theta\rangle)\right], \qquad (3.15) \\ \phi &= \arctan\left[\frac{e^{-L\langle\lambda\rangle}\sin(L\langle\theta\rangle)}{1 + e^{-L\langle\lambda\rangle}\cos(L\langle\theta\rangle)}\right], \qquad (3.16) \end{aligned}$$

where $\mathrm{Li}_n(r,\theta)$ is the real part of the polylogarithm $\mathrm{Li}_n(re^{\mathrm{i}\theta})$ and $Cl_n(\theta)$ is Clausen's function [19]. An important observation of [14] is that for $L\langle\theta\rangle = \pi$ the fermions acquire periodic boundary conditions and the theory resembles the bosonic $\mathrm{O}(N)$ vector model discussed previously. In this case the expectation value of λ is given by

$$L\langle\lambda\rangle = 2\ln\tau \tag{3.17}$$

and, from eq. (3.15), the free-energy is

$$\frac{C(G_*)}{N} = -\frac{8}{5}\frac{\zeta(3)}{2\pi}\,, \tag{3.18}$$

i.e. minus twice the free-energy density of the $\mathrm{O}(N)$ vector model. The interpretation of a negative free-energy density may not be clear when one considers the model above by itself. It is entirely justifiable, however, when the model is viewed as part of a larger theory containing both fermions and bosons as we discuss in the next section.

4. Stationary trajectories and the free-energy density

The models described in section 3 are typical examples of strong-coupling dynamics in QFT. The low-energy critical theory is unreachable by a weak coupling expansion around the UV-critical point and one needs to know the full analytic dependence of physical quantities on the coupling constant to study both regimes. Most importantly, the spectrum of the theory changes in an uncontrollable way.

To understand the difference between the massless degrees of freedom coupled to the different critical regimes of the O(N) model, we shall exploit the following idea. We shall construct stationary flows which interpolate between critical theories with different field contents, one of which is the free-field theory of N massless scalars. Since the free-energy density remains invariant along stationary flows, all such theories will have the same free-energy density eq. (3.11). These theories may be interpreted as interacting theories of massive scalar fields. Then, the condition for stationary free-energy density implies definite relations between the masses. For a set of special values of the masses, the resulting theory is seen to be related to the strongly coupled critical theory eq. (3.10).

Consider a theory in three dimensions with nine O($N/4$) bosonic fields, $\vec{\phi}_I$ ($I = 1, \ldots, 9$), and two U($N/4$) fermionic fields $\vec{\psi}_J$ ($J = 1, \ldots, 4$). These fields have masses m_I, ($I = 1, \ldots, 9$) and m'_J, ($J = 1, 2$), respectively. The quadratic part of the lagrangean density reads

$$\mathcal{L}_0 = \frac{1}{2}\sum_{I=1}^{9} \vec{\phi}_I \cdot (-\partial^2 + m_I^2)\vec{\phi}_I - \sum_{J=1}^{2} \bar{\psi}_J^\alpha(\gamma_\mu \partial^\mu + m'_J)\psi_J^\alpha \, , \tag{4.1}$$

where $\alpha = 1, 2, \ldots, N/4$. These fields interact, but we will not specify the interaction lagrangean. Furthermore, we may impose a fixed total fermion number constraint as it was discussed in section 3. This is quite natural since the total fermion number constraint is equivalent to the presence of a background gauge field [22, 23]. Hence, the theory in eq. (4.1) may be viewed as a three-dimensional massive gauge theory with bosons and fermions. The full lagrangean density is

$$\mathcal{L} = \mathcal{L}_0 + V_{\mathrm{int}}(\vec{\phi}_I, \psi_J^\alpha) \, , \tag{4.2}$$

where the interaction potential V_{int} is a function of O($N/4$) invariants made out of the bosons and fermionic currents.

Each bosonic field has free-energy in the saddle point approximation

$$F_b^{(I)} = \frac{N}{8} \operatorname{Tr}\ln(-\partial^2 + m_I^2) \, , \qquad I = 1, \ldots, 9 \, . \tag{4.3}$$

The corresponding expression for a fermionic field is

$$F_f^{(J)} = -\frac{N}{4} \operatorname{Tr}\ln(-\partial^2 + (m'_J)^2) \, , \qquad J = 1, 2 \, . \tag{4.4}$$

The total free-energy of the system is

$$F = \sum_{I=1}^{9} F_b^{(I)} + \sum_{J=1}^{2} F_f^{(J)} \, . \tag{4.5}$$

In the slab geometry bosons acquire periodic boundary conditions along the finite dimension $p_0 = 2\pi n/L, n = 0, \pm 1, \pm 2, \ldots$ Furthermore, due to the total fermion number constraint [14], or equivalently due to the some special configuration of the background gauge field [23], we can arrange so that fermions acquire periodic boundary conditions as well. Then, using the identity

$$\sum_{n=-\infty}^{\infty} \ln(\omega_n^2+|\vec{p}|^2+m^2) = 2\ln\left(1-e^{-L\sqrt{|\vec{p}|^2+m^2}}\right)+L\int\frac{dp_0}{2\pi}\ln(p_0^2+|\vec{p}|^2+m^2)\,, \quad (4.6)$$

we obtain

$$F_b^{(I)} = \frac{N}{4L}\int\frac{d^2\vec{p}}{(2\pi)^2}\ln\left(1-e^{-L\sqrt{|\vec{p}|^2+m_I^2}}\right)+\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\ln(p^2+m_I^2) \quad (4.7)$$

and similarly for the fermions. These expressions are divergent. We will subtract the UV divergence à la Bogoliubov. To this end, we expand

$$\ln(p^2+m_I^2) = \ln p^2+\frac{m_I^2}{p^2}+o(m_I^4) \quad (4.8)$$

and replace

$$\ln(p^2+m_I^2) \to \ln(p^2+m_I^2)-\ln p^2-\frac{m_I^2}{p^2}\,. \quad (4.9)$$

The resulting expression is finite and given by

$$\frac{8\pi L^3 F_{b\,R}^{(I)}}{N} = \mathrm{Li}_3(e^{-m_I L})-\ln(e^{-m_I L})\mathrm{Li}_2(e^{-m_I L})-\frac{1}{6}(m_I L)^3\,. \quad (4.10)$$

Corrections to this expression may come from the interactive part of the lagrangean eq. (4.2) and are in general polynomials in the masses. As long as we are interested in stationary trajectories, we will consider only marginal couplings which are trinomials in the masses. We will add appropriate trinomials to obtain a free-energy which is invariant along a certain trajectory in the space of couplings.

Pick a RG trajectory ($\beta(g)=0$) parametrized by t.[3] Along this trajectory, we demand that the function

$$C = 2\pi\frac{L^3}{N}\left(\sum_{I=1}^{9}F_{b\,R}^{(I)}+\sum_{J=1}^{2}F_{f\,R}^{(I)}\right)+L^3\sum_{i,j,k}\alpha_{ijk}m_i m_j m_k \quad (4.11)$$

be invariant. It turns out that a convenient choice is

$$\sum_{i,j,k}\alpha_{ijk}m_i m_j m_k = \sum_{I=1}^{4}\left(\frac{1}{3}m_I^3-\frac{1}{2}m_I m_{I+4}^2-\frac{1}{2}m_I^2 m_{I+4}+\frac{1}{3}m_{I+4}^3\right). \quad (4.12)$$

Defining

$$x_I = e^{-m_I L} \quad (I=1,\ldots,9)\,, \qquad y_J = e^{-m'_J L} \quad (J=1,2) \quad (4.13)$$

[3]This plays now the role of the new parameter t' discussed in the introduction.

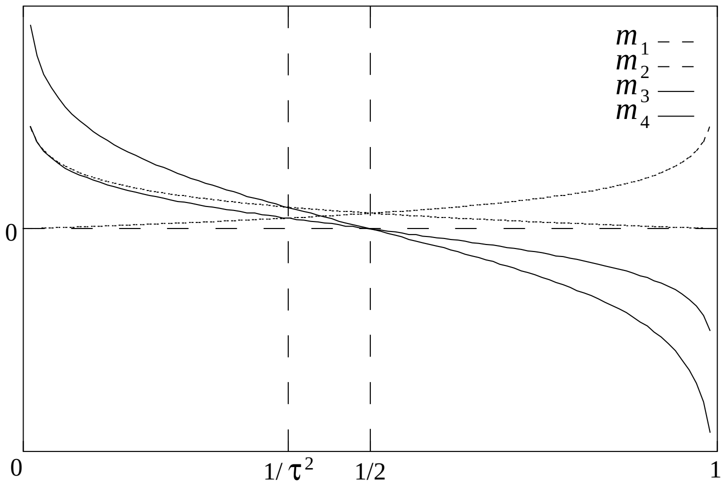

Figure 1: Flow of masses for $0 < t < 1$. Some masses become negative for $t > 1/2$. Pairs meet at $t = 1/\tau^2$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean, and at $t = 1/2$.

and setting

$$x_1 = x_2 = x_3 = x_4 = t\,, \qquad x_5 = x_6 = x_7 = x_8 = 1 - t\,, \qquad x_9 = \frac{t^2}{(1-t)^2}$$
$$y_1 = y_2 = \frac{t}{1-t} \tag{4.14}$$

we obtain

$$\frac{dC}{dt} = 0\,, \tag{4.15}$$

which is the result of non-trivial polylogarithm identities (see appendix).

Next, we discuss the behavior of the theory along the trajectory. The flow of the masses is shown in figure 1. To consider theories with real, non-negative masses, the parameter t must be restricted to the interval

$$0 \leq t \leq \frac{1}{2}\,. \tag{4.16}$$

Along the trajectory, $0 < t < 1/2$, we have $\mathrm{O}(N/4)$ symmetry. At the two ends, $t = 0, 1/2$, we obtain enhanced mass degeneracy. There is also a special point in the interval $(0, 1/2)$ where mass degeneracy is enhanced and the symmetry group becomes $\mathrm{O}(5N/4)$.

As $t \to 0$, we have $x_1, x_2, x_3, x_4, x_9 \to 0$, as well as $y_1, y_2 \to 0$. Therefore, the bosonic fields $\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3, \vec{\phi}_4, \vec{\phi}_9$ and all four fermionic fields become infinitely massive and decouple. The other four bosonic fields have $x_I \to 1$ ($I = 5, \ldots, 8$), so they all become massless. Therefore, in this limit we obtain $4(N/4) = N$ free massless bosonic fields. Eq. (4.11) becomes

$$C = C(0) = \mathrm{Li}_3(1) \tag{4.17}$$

and the underlying theory which flows to the $t = 0$ point of the above stationary RG-flow is the free theory of N massless bosons.

There is an interesting point in the interval $0 \leq t \leq 1/2$ at which the masses are divided into two sets, all masses in the same set converging to the same value. Let τ be the "golden mean"

$$\tau^2 - \tau - 1 = 0\,, \qquad \tau = \frac{1+\sqrt{5}}{2}\,. \tag{4.18}$$

At $t = 1/\tau^2 = 2 - \tau$, we have

$$\begin{gathered} x_1 = x_2 = x_3 = x_4 = x_9 = 2 - \tau\,, \qquad x_5 = x_6 = x_7 = x_8 = \tau - 1\,, \\ y_1 = y_2 = \frac{1}{\tau^2 - 1} = \frac{1}{\tau} = \tau - 1\,. \end{gathered} \tag{4.19}$$

At this point, eq. (4.11) becomes

$$C = \frac{5}{4}\left(\mathrm{Li}_3(2-\tau) - \ln(2-\tau)\mathrm{Li}_2(2-\tau) + \frac{1}{6}\ln^3(2-\tau)\right), \tag{4.20}$$

However, this is nothing but 5/4 times eq. (3.10), the latter giving the free-energy of the UV critical point of the O(N) vector model. This means that the underlying theory flowing to the $t = 2 - \tau$ point of the stationary flow above corresponds to 5/4 copies of the O(N) vector model. Consequently, the degrees of freedom coupled to the non-trivial UV critical point of the O(N) vector model are 4/5 times the degrees of freedom coupled to the free massless theory of N scalars, q.e.d.

5. Discussion and outlook

In this work, we addressed the issue of non-perturbative RG flows connecting weakly and strongly coupled fixed-points of three-dimensional QFTs. A quantity which encodes important information for such fixed points is the free-energy density. We briefly discussed some of its properties in section 3 and analyzed some old [12], and more recent [14] exact results for the free-energy density of non-trivial, strongly coupled three-dimensional CFTs in section 4. We presented our main result in section 5, where we provided an explicit example of a stationary trajectory interpolating between a class of critical three-dimensional QFTs. The crucial observation is that the interactive critical theory corresponding to a special point of the trajectory is related to the non-trivial critical point of the three-dimensional O(N) vector model. This way, we provided a physical picture for the fact that the latter critical theory seems to involve 4/5 times the number of massless degrees of freedom of the free-field theory of N massless scalars. Our arguments are based on the observation that the stationary points of the quantity $C(t)$ correspond to possible critical points. The introduction of the finite length L was also essential in order to obtain more than one non-trivial stationary points for $C(t)$.

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Along the stationary flow, we have a $O(N/4)$ symmetry. At the two ends of the flow (see figure 1), we have enhanced mass degeneracy. At one end, we obtain $4(N/4) = N$ free massless bosons; at the other end, we obtain an enhanced symmetry, $O(5N/4)$ and the model rolls to a non-trivial conformal field theory. In obtaining the stationary trajectory of eqs. (4.11) and (4.15) it was essential that the fermions in the lagrangean eq. (4.1) acquire periodic boundary conditions. This is not unusual for fermions in gauge field backgrounds, as the zeroth component of the gauge field plays the role of an imaginary chemical potential. Therefore, it is possible that for certain configurations of the background gauge field the fermions acquire a zero mode and are effectively bozonized contributing, however, a negative free-energy density [14, 23]. Although the issue regarding the physical interpretation of such a bosonization is not clearly settled as yet [24], we have explicitly demonstrated its relevance to studies of non-perturbative RG flows.

Our approach to studying the free-energy density of critical theories connected via non-perturbative RG-flows could in principle be applied to other interesting cases, such as the $\mathcal{N} = 4$ SYM$_4$, in connection with the AdS/CFT correspondence. There, it is found that the free-energy density of the $\mathcal{N} = 4$ SYM is $3/4\times$ the free massless theory. This is a puzzle with no apparent resolution. Our results suggest that in order to address this issue, one might try to construct a stationary flow, within the massless-free theory and see whether at some point it becomes equal to $4/3\times$ the free energy of a non-trivial CFT. Such an extension of our results is expected to be more complicated, as the corresponding formulae for the free-energy density of massive four-dimensional theories involve infinite sums of polylogarithms [25]. A further complication comes from the fact that the beta-function of $\mathcal{N} = 4$ SYM$_4$ is identically zero. Therefore, neither the RG eq. (2.3), nor any kind of weak-coupling expansion are sufficient to studying the coupling constant dependence of $C(g, l)$. In this respect, all efforts towards understanding the non-perturbative nature of $\mathcal{N} = 4$ SYM$_4$ are worthwhile.

A. Some polylogarithmic indentities

Here we present certain polylogarithm identities which we used in order to derive various results earlier. For more details see [19].

The nth-order polylogarithm is defined by

$$\mathrm{Li}_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^n}\,. \tag{A.1}$$

We have $\mathrm{Li}_n(1) = \zeta(n)$.

The dilogarithm obeys the following identities:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \mathrm{Li}_2(1) - \ln x \ln(1-x)\,, \tag{A.2}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\mathrm{Li}_2(x^2)\,, \tag{A.3}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{-x}{1-x}\right) = -\frac{1}{2}\ln^2(1-x)\,. \tag{A.4}$$

The golden mean plays a special role in these identities. Let $\tau = \frac{1+\sqrt{5}}{2}$. Setting $x = 1/\tau$ in the first two identities (eqs. (A.2) and (A.3)), we obtain

$$\frac{3}{2}\mathrm{Li}_2\left(\frac{1}{\tau^2}\right) - \mathrm{Li}_2\left(-\frac{1}{\tau}\right) = \mathrm{Li}_2(1) - \frac{1}{2}\ln\tau^2\,, \tag{A.5}$$

where we used $\tau^2 - \tau - 1 = 0$. Setting $x = 1/\tau^2$ in the third identity (eq. (A.4)), we obtain

$$\mathrm{Li}_2\left(\frac{1}{\tau^2}\right) - \mathrm{Li}_2\left(-\frac{1}{\tau}\right) = -\frac{1}{8}\ln^2\tau^2\,. \tag{A.6}$$

Combining eqs. (A.5) and (A.6), we finally obtain

$$\mathrm{Li}_2\left(\frac{1}{\tau^2}\right) = \frac{2}{5}\mathrm{Li}_2(1) - \frac{1}{4}\ln^2\tau^2\,, \tag{A.7}$$

i.e. the dilogarithm can be expressed in terms of elementary functions at the special point $x = 1/\tau^2$.

Similarly, for the trilogarithm, the following identities hold:

$$\begin{aligned}\mathrm{Li}_3(x) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3\left(\frac{-x}{1-x}\right) &= \mathrm{Li}_3(1) + \mathrm{Li}_2(1)\ln(1-x) - \\ &\quad - \frac{1}{2}\ln x \ln^2(1-x) + \frac{1}{6}\ln^3(1-x)\,,\end{aligned} \tag{A.8}$$

$$\mathrm{Li}_3(x) + \mathrm{Li}_3(-x) = \frac{1}{4}\mathrm{Li}_3(x^2)\,. \tag{A.9}$$

Again, the golden mean plays a special role. To see this, set $x = 1/\tau^2$ in the first identity (eq. (A.8))

$$\mathrm{Li}_3\left(\frac{1}{\tau^2}\right) + \mathrm{Li}_3\left(\frac{1}{\tau}\right) + \mathrm{Li}_3\left(-\frac{1}{\tau}\right) = \mathrm{Li}_3(1) + \frac{1}{2}\mathrm{Li}_2(1)\ln\tau^2 - \frac{5}{48}\ln^3\tau^2\,. \tag{A.10}$$

Using the second identity (eq. (A.9)), we obtain

$$\mathrm{Li}_3\left(\frac{1}{\tau^2}\right) = \frac{4}{5}\mathrm{Li}_3(1) - \frac{2}{5}\mathrm{Li}_2(1)\ln\tau^2 - \frac{1}{12}\ln^3\tau^2\,, \tag{A.11}$$

which, in view of eq. (A.7), can be written as

$$\mathrm{Li}_3\left(\frac{1}{\tau^2}\right) + \mathrm{Li}_2\left(\frac{1}{\tau^2}\right)\ln\tau^2 - \frac{1}{6}\ln^3\tau^2 = \frac{4}{5}\mathrm{Li}_3(1)\,. \tag{A.12}$$

Introducing the function

$$\mathcal{F}(x) = \mathrm{Li}_3(x) - \mathrm{Li}_2(x)\ln x + \frac{1}{6}\ln^3 x \tag{A.13}$$

we can write eq. (A.12) as (*cf.* eq. (3.10))

$$\mathcal{F}\left(\frac{1}{\tau^2}\right) = \frac{4}{5}\mathcal{F}(1) = \frac{4}{5}\zeta(3)\,. \tag{A.14}$$

To show that the function eq. (4.11) is invariant under the flow eq. (4.14), set $x = t$ in eq. (A.8) and $x = t/(1-t)$ in eq. (A.9). Combining the two equations, we obtain

$$\begin{aligned} \mathrm{Li}_3(t) + \mathrm{Li}_3(1-t) + \frac{1}{4}\mathrm{Li}_3\left(\frac{t^2}{(1-t)^2}\right) - \mathrm{Li}_3\left(\frac{t}{1-t}\right) &= \\ = \mathrm{Li}_3(1) + \mathrm{Li}_2(1)\ln(1-t) - -\frac{1}{2}\ln t\ln^2(1-t) + \frac{1}{6}\ln^3(1-t)\,. \end{aligned} \tag{A.15}$$

Now define the symmetric function (cf. eq. (4.12))

$$\mathcal{V}(x,y) = \frac{1}{3}\ln^3 x - \frac{1}{2}\ln^2 x\ln y - \frac{1}{2}\ln x\ln^2 y + \frac{1}{3}\ln^3 y\,. \tag{A.16}$$

After some algebra, we obtain

$$\mathcal{C} = \frac{1}{4}\sum_{I=1}^{9}\mathcal{F}(x_I) - \frac{1}{2}\sum_{J=1}^{2}\mathcal{F}(y_J) - \sum_{I=1}^{4}\mathcal{V}(x_I, x_{I+4}) = \mathrm{Li}_3(1)\,, \tag{A.17}$$

where the x_i $(i = 1, \ldots, 9)$, y_1, y_2 are given by eq. (4.14). eq. (A.17) implies stationary flow (cf. eq. (4.15)),

$$\frac{d\mathcal{C}}{dt} = 0\,. \tag{A.18}$$

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